EE 505
Lecture 10

- Statistical Circuit Modeling

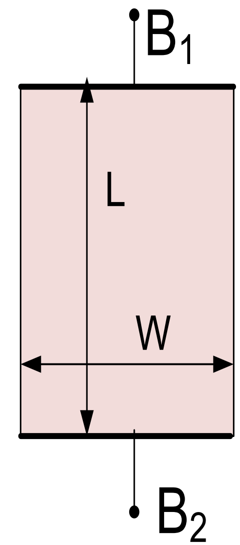
Review from previous lecture:

Summary of Results

Structure	Nominal Resistance	Standard Deviation	Normalized Standard Deviation
R	R_N	σ_{R_R}	$\sigma_{\frac{R_R}{R_N}}$
Ser nR	nR_N	$\sqrt{n}\sigma_{R_R}$	$\frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$
Par nR	$\frac{R_N}{n}$	$\frac{1}{n^{3/2}}\sigma_{R_R}$	$\frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$
Ser 2R	$2R_N$	$\sqrt{2}\sigma_{R_R}$	$\sigma_{\frac{R_R}{R_N}}/\sqrt{2}$
Par 2R	$\frac{R_N}{2}$	$\sigma_{R_R}/\sqrt{8}$	$\sigma_{\frac{R_R}{R_N}}/\sqrt{2}$
Ser 4R	$4R_N$	$2\sigma_{R_R}$	$\sigma_{\frac{R_R}{R_N}}/2$
Par 4R	$\frac{R_N}{4}$	$\sigma_{R_R}/8$	$\sigma_{\frac{R_R}{R_N}}/2$
Par/Ser 4R	R_N	$\sigma_{R_R}/2$	$\sigma_{\frac{R_R}{R_N}}/2$

Review from previous lecture:

Consider a resistor of width W and length L



$$\sigma_R^2 = \left(\frac{L}{W}\right)^2 \bullet \frac{\sigma_{REF}^2}{W \bullet L} = \sigma_{REF}^2 \bullet \frac{L}{W^3}$$

A=W•L

Note σ_R is dependent on resistance value

Consider now the normalized resistance $\frac{R}{R_N}$

where $R_N = R_{\square N}\frac{L}{W}$

It follows that

$$\sigma_{\frac{R}{R_N}}^2 = \left(\frac{1}{R_N^2}\right)\left(\sigma_{REF}^2 \frac{L}{W^3}\right) = \left(\frac{W^2}{R_{\square N}^2 L^2}\right)\left(\sigma_{REF}^2 \frac{L}{W^3}\right) = \left(\frac{1}{WL}\right)\left[\frac{\sigma_{REF}^2}{R_{\square N}^2}\right]$$

The term on the right in [ ] is the ratio of two process parameters so define the process parameter A_R by the expression $A_R = \frac{\sigma_{REF}}{R_{\square N}}$

A_R is more convenient to use than both σ_{REF} and $R_{\square N}$

Thus the normalized resistance is given by the expression

$$\sigma_{\frac{R}{R_N}}^2 = \frac{A_R^2}{WL} = \frac{A_R^2}{A}$$

Note $\sigma_{R/RN}$ is not dependent on resistance value

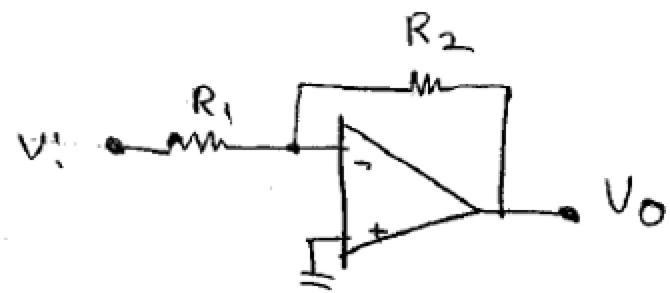
Will term A_R the "Pelgrom parameter" (though Pelgrom only presented results for MOS devices)

Amplifier Gain Accuracy

Review from previous lecture:

$$\theta = K - \left(K + \sum_{i=1}^{K} \frac{R_{R2i}}{R_0} - \frac{K R_{R1}}{R_0} + \theta \right)$$

$-$ $+$



$$\theta \simeq \sum_{i=1}^{K} \frac{R_{R2i}}{R_0} - K \frac{R_{R1}}{R_0}$$

θ is the gain error

$$\sigma_\theta^2 = K \sigma^2_{\frac{R_R}{R_N}} + K^2 \sigma^2_{\frac{R_R}{R_N}}$$

$$\boxed{\sigma_\theta = \sigma_{\frac{R_R}{R_N}} \sqrt{K + K^2}}$$

Note: K is simply the nominal magnitude of the dc gain

If $K = 1$ $\quad \sigma_\theta = \sigma_{\frac{R_R}{R_N}} \sqrt{2}$

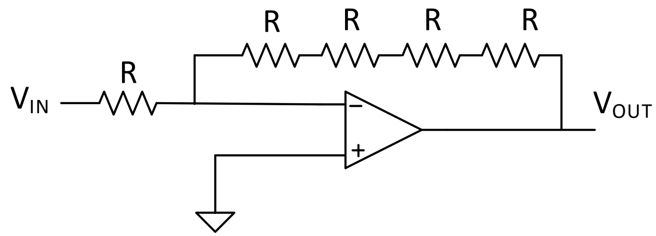
$K = 10$ $\quad \sigma_\theta = \sigma_{\frac{R_R}{R_N}} \sqrt{110}$

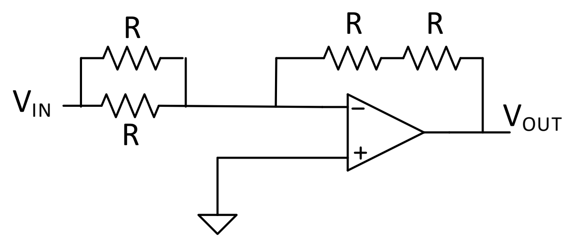
$$\sigma_\theta \simeq 10.5\, \sigma_{\frac{R_R}{R_N}}$$

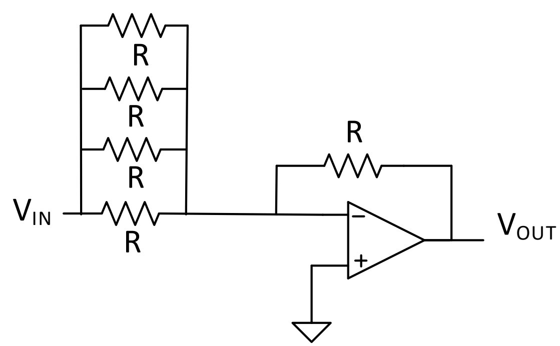
Amplifier Gain Accuracy

Review from previous lecture:

Many different ways to achieve a given gain with a given resistor area

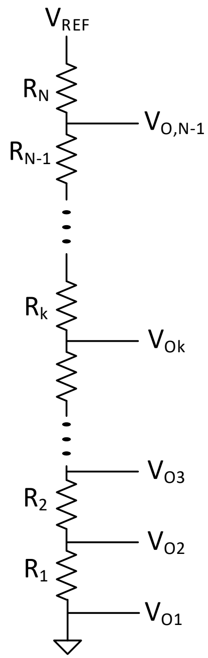






Which will have the best yield?

String DAC Statistical Performance



$0 \le k \le N-1$

- INL is of considerable interest
- INL=Max($|INL_k|$), $0<k<N-1$
- INL is difficult to characterize analytically so will focus on INL_k

Assume resistors are uncorrelated RVs but identically distributed, typically zero mean Gaussian

Consider $INL_k = V_{OUT}(k) - V_{FIT}(k)$

$$V_{OUT}(k) = \begin{cases} 0 & k=0 \\ \dfrac{\sum_{j=1}^{k} R_j}{\sum_{j=1}^{N} R_j} V_{REF} & 1 \le k \le N-1 \end{cases}$$

$$V_{FIT}(k) = \frac{\mathrm{k}}{\mathrm{N}\text{-}1} \frac{\sum_{j=1}^{N-1} R_j}{\sum_{j=1}^{N} R_j} V_{REF} \qquad 0 \le \mathrm{k} \le \mathrm{N}\text{-}1$$

String DAC Statistical Performance

$$INL_k = \frac{\left(\dfrac{\sum_{j=1}^{k} R_j}{\sum_{j=1}^{N} R_j} - \dfrac{k}{N-1}\dfrac{\sum_{j=1}^{N-1} R_j}{\sum_{j=1}^{N} R_j}\right) V_{REF}}{\dfrac{V_{REF}}{2^n}} \qquad 1 \le k \le N-1$$

$$INL_k = \frac{\sum_{j=1}^{k} R_j - \dfrac{k}{N-1}\sum_{j=1}^{N-1} R_j}{\sum_{j=1}^{N} R_j} 2^n \qquad 1 \le k \le N-1$$

$$INL_k = \frac{\sum_{j=1}^{k} R_j - \dfrac{k}{N-1}\sum_{j=1}^{k} R_j - \dfrac{k}{N-1}\sum_{j=k+1}^{N-1} R_j}{\sum_{j=1}^{N} R_j} 2^n \qquad 1 \le k \le N-1$$

$$INL_k = \frac{\sum_{j=1}^{k} R_j \left(1 - \dfrac{k}{N-1}\right) - \dfrac{k}{N-1}\sum_{j=k+1}^{N-1} R_j}{\sum_{j=1}^{N} R_j} 2^n \qquad 1 \le k \le N-1$$

Let $R_j = R_{NOM} + R_{Rj}$

String DAC Statistical Performance

$$INL_k = \frac{\left[\sum_{j=1}^{k} R_{NOM}\left(1-\frac{k}{N-1}\right)-\frac{k}{N-1}\sum_{j=k+1}^{N-1} R_{NOM}\right]+\sum_{j=1}^{k} R_{Rj}\left(1-\frac{k}{N-1}\right)-\frac{k}{N-1}\sum_{j=k+1}^{N-1} R_{Rj}}{\sum_{j=1}^{N} R_{NOM}+\sum_{j=1}^{N} R_{Rj}} 2^n \qquad 1 \le k \le N-1$$

$$INL_k = \frac{R_{NOM}\left[k\left(1-\frac{k}{N-1}\right)-\frac{k}{N-1}(N-k-1)\right]+\sum_{j=1}^{k} R_{Rj}\left(1-\frac{k}{N-1}\right)-\frac{k}{N-1}\sum_{j=k+1}^{N-1} R_{Rj}}{NR_{NOM}+\sum_{i=1}^{N} R_{Rj}} 2^n \qquad 1 \le k \le N-1$$

$$INL_k = \frac{2^n}{NR_{NOM}} \frac{\sum_{j=1}^{k} R_{Rj}\left(1-\frac{k}{N-1}\right)-\frac{k}{N-1}\sum_{j=k+1}^{N-1} R_{Rj}}{1+\frac{1}{NR_{NOM}}\sum_{j=1}^{N} R_{Rj}} \qquad 1 \le k \le N-1$$

If we do a Taylor's series expansion of the reciprocal of the denominator and eliminate second-order and higher terms it follows that

$$INL_k = \frac{1}{R_{NOM}}\left[\sum_{j=1}^{k} R_{Rj}\left(1-\frac{k}{N-1}\right)-\frac{k}{N-1}\sum_{j=k+1}^{N-1} R_{Rj}\right]\left[1-\frac{1}{NR_{NOM}}\sum_{j=1}^{N} R_{Rj}\right] \qquad 1 \le k \le N-1$$

$$INL_k = \frac{1}{R_{NOM}}\left[\sum_{j=1}^{k} R_{Rj}\left(1-\frac{k}{N-1}\right)-\frac{k}{N-1}\sum_{j=k+1}^{N-1} R_{Rj}\right] \qquad 1 \le k \le N-1$$

Note that INK_k is a zero-mean multivariate Gaussian distribution

String DAC Statistical Performance

$$INL_k = \frac{1}{R_{NOM}}\left[\sum_{j=1}^{k} R_{Rj}\left(1-\frac{k}{N-1}\right)-\frac{k}{N-1}\sum_{j=k+1}^{N-1} R_{Rj}\right] \qquad 1 \le k \le N-1$$

Since the resistors are identically distributed and the coefficients are not a function of the index j, it follows that

$$\sigma^2_{INLk} = \sigma^2_{\frac{R_R}{R_{NOM}}}\left[\sum_{j=1}^{k}\left(1-\frac{k}{N-1}\right)^2+\sum_{j=k+1}^{N-1}\left(\frac{k}{N-1}\right)^2\right] \qquad 1 \le k \le N-1$$

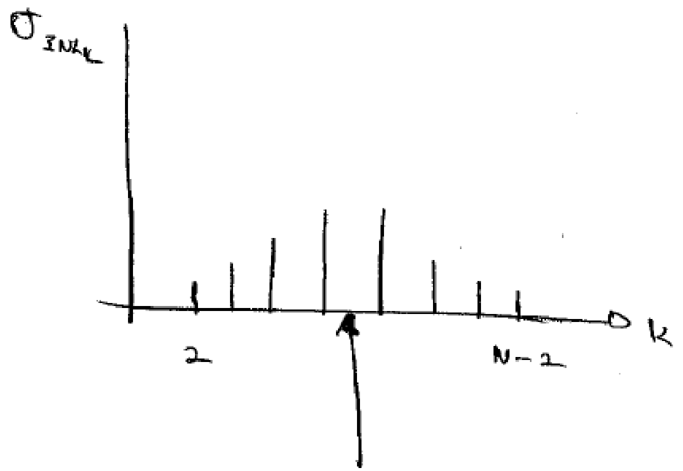
Since the index in the sum does not appear in the arguments, this simplifies to

$$\sigma_{INLk} = \sigma_{\frac{R_R}{R_{NOM}}}\sqrt{\frac{(N-1-k)k}{N-1}} \qquad 1 \le k \le N-1$$

Note there is a nice closed-form expression for the INL_k for a string DAC !!

String DAC Statistical Performance

INL_k assumes a maximum variance at mid-code



max of σ_{INLk}

occurs at $k \simeq \frac{N}{2}$

$$\sigma_{INLk\,\max} = \sigma_{\frac{R_R}{R_{NOM}}} \frac{\sqrt{N}}{2}$$

String DAC Statistical Performance

How about statistics for the INL?

$$INL = \max_k |INL_k|$$

$$INL_k = \sum_{i=1}^{k-1} \frac{R_{R_i}}{R_N}\left(\frac{N-k}{N-1}\right) - \sum_{i=k}^{N-1} \frac{R_{R_i}}{R_N}\left(\frac{k-1}{N-1}\right)$$

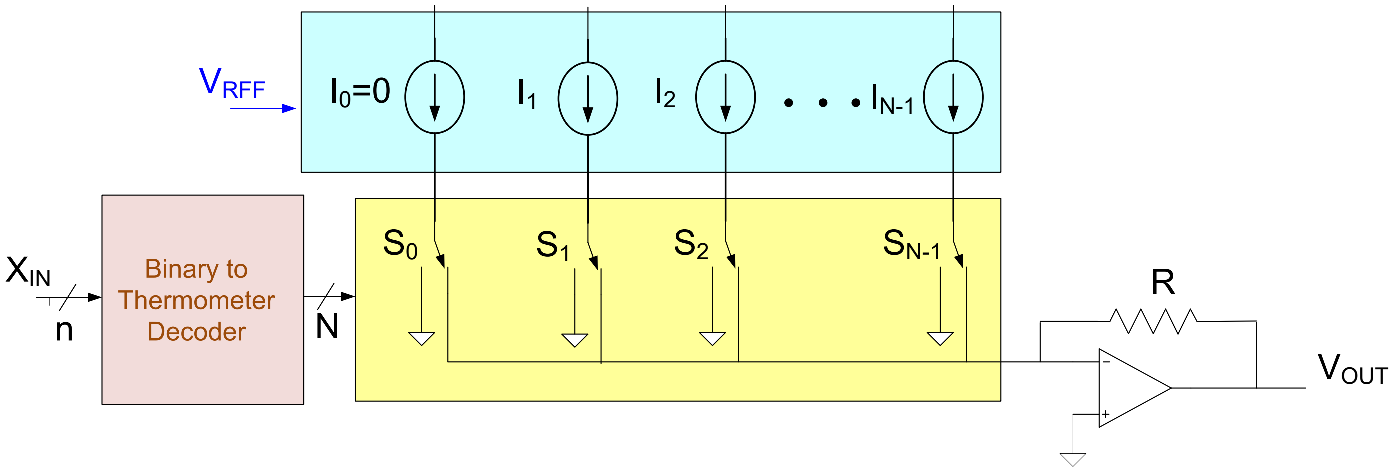
INL is an order statistic

Distribution functions for order statistics are very complicated and closed form solutions do not exist

INL is not zero-mean and not Gaussian

Current Steering DAC Statistical Characterization

Unary weighted



Assume unary current source array and define I_0=0

$$V_{OUT}(k) = -R\sum_{j=0}^{k-1} I_j \qquad 1 \le k \le N$$

For notational convenience will normalize by –R to obtain

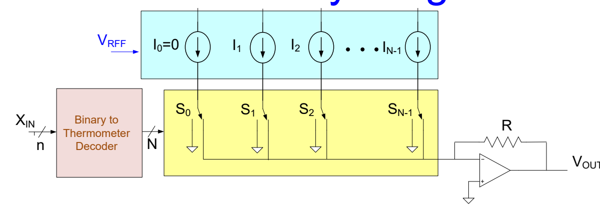
$$I_{OUTX}(k) = \sum_{i=0}^{k-1} I_i \qquad 1 \le k \le N$$

Assume current sources are random variables with identical distributions

$$I_j = I_{NOM} + I_{Rj} \qquad I_{Rj} \propto N(0, \sigma_I)$$

Current Steering DAC Statistical Characterization

Unary weighted



$$INL_k(k)=\frac{\sum_{j=0}^{k-1} I_j - I_{FIT}(k)}{I_{NOM}}$$

$1 \leq k \leq N$

$$I_{FIT}(k)=\frac{k-1}{N-1}\left(\sum_{j=1}^{N-1} I_j\right)$$

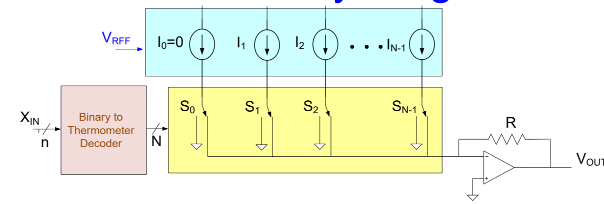
$1 \leq k \leq N$

$$INL_k(k)=\frac{\sum_{j=1}^{k-1} I_j - \frac{k-1}{N-1}\left(\sum_{j=1}^{N-1} I_j\right)}{I_{NOM}}$$

$$INL_k=\frac{\sum_{i=1}^{k-1}\left(1-\frac{k-1}{N-1}\right)I_i - \frac{k-1}{N-1}\sum_{i=k}^{N-1} I_i}{I_{NOM}}$$

Current Steering DAC Statistical Characterization

Unary weighted



$$INL_k = \frac{\sum_{i=1}^{k-1}\left(1-\frac{k-1}{N-1}\right)I_i - \frac{k-1}{N-1}\sum_{i=k}^{N-1}I_i}{I_{NOM}}$$

Model the current sources as $I_j = I_{NOM} + I_{Rj}$

$$INL_k = \frac{\sum_{i=1}^{k-1}\left(1-\frac{k-1}{N-1}\right)\left(I_{NOM}+I_{Rk}\right) - \frac{k-1}{N-1}\sum_{i=k}^{N-1}\left(I_{NOM}+I_{Rk}\right)}{I_{NOM}}$$

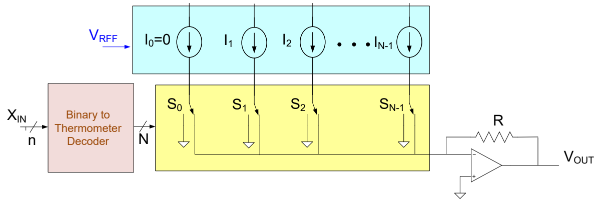
It can be shown that the nominal part cancels, thus

$$INL_k = \sum_{i=1}^{k-1}\left(\frac{N-k}{N-1}\right)\left(\frac{I_{Rk}}{I_{NOM}}\right) - \frac{k-1}{N-1}\sum_{i=k}^{N-1}\left(\frac{I_{Rk}}{I_{NOM}}\right)$$

This is a sum of uncorrelated random variables

Current Steering DAC Statistical Characterization

The variance of I_{NKk} can be readily calculated



$$\mathrm{I_j = I_N + I_{Rj}}$$

$$\sigma^2_{INL_k} = \sum_{i=1}^{k-1}\left(\frac{N-k}{N-1}\right)^2 \sigma^2_{\frac{I_{Rk}}{I_{NOM}}} + \left(\frac{k-1}{N-1}\right)^2 \sum_{i=k}^{N-1} \sigma^2_{\frac{I_{Rk}}{I_{NOM}}}$$

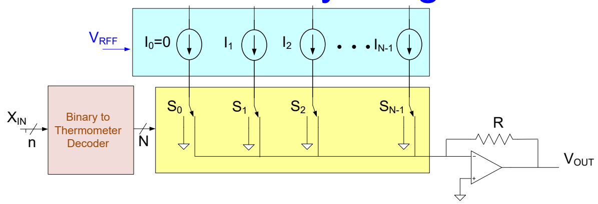
$$\sigma^2_{INL_k} = \left[(k-1)\left(\frac{N-k}{N-1}\right)^2 + (N-k)\left(\frac{k-1}{N-1}\right)^2\right]\sigma^2_{\frac{I_{Rk}}{I_{NOM}}}$$

This simplifies to

$$\sigma^2_{INL_k} = \frac{(k-1)(N-k)}{(N-1)}\sigma^2_{\frac{I_{Rk}}{I_{NOM}}}$$

Current Steering DAC Statistical Characterization

Unary weighted



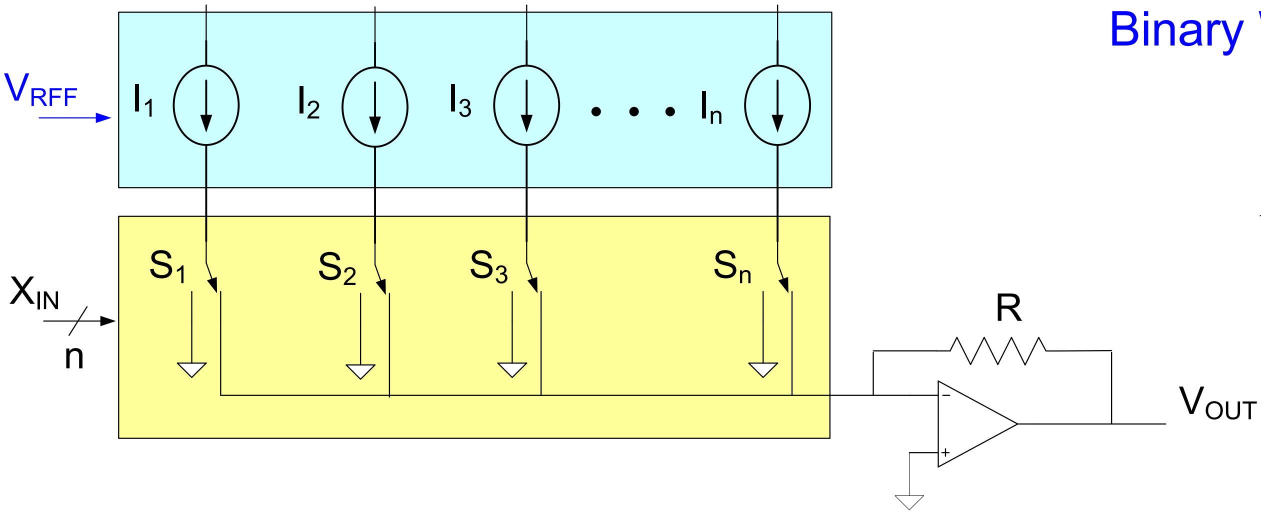
As for the string DAC, the maximum INL_k occurs near mid-code at about k=N/2 thus

$$I_j = I_N + I_{Rj}$$

$$\sigma_{INL_{k-MAX}} = \sigma_{\frac{I_R}{I_{NOM}}} \left[\frac{\sqrt{N}}{2} \right]$$

And, as for the string DAC, the INL is an order statistic and thus a closed-form solution does not exist

Current Steering DAC Statistical Characterization



Binary Weighted

Ideally

$$I_j = 2^{j-1} I_{NOM}$$

$$1 \le j \le n\text{-}1$$

The structure looks about the same as for the unary structure but now the current sources are binary weighted

$$V_{OUT}(\mathbf{b}) = -R\sum_{j=0}^{n} b_i I_j \qquad \mathbf{b} = \langle b_n, b_{n-1} \ldots b_1 \rangle$$

Define the decimal equivalent of b, k_b , by

$$k_{\mathbf{b}} = \sum_{j=1}^{n} b_j 2^{j-1}$$

For notational convenience will normalize by –R to obtain

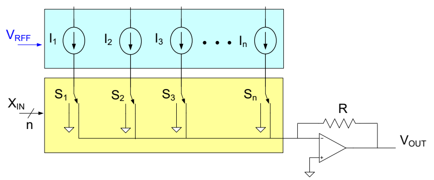
$$I_{OUTX}(\mathbf{b}) = \sum_{i=1}^{n} b_i I_i \qquad \text{for } <0,0,\ldots 0> \ \le \mathbf{b} \le \ <1,1,\ldots 1>$$

Current Steering DAC Statistical Characterization

Binary Weighted

$$I_{FIT}(\mathbf{b})=\frac{k_{\mathbf{b}}}{N-1}\sum_{i=1}^{n} I_i$$

$0 \le k_b \le N\text{-}1$



Thus

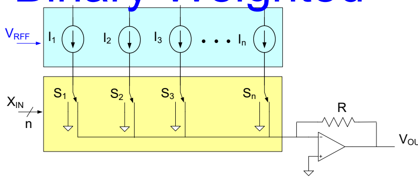
$$INL_{\mathbf{k}}(\mathbf{b})=\frac{I_{OUTX}(\mathbf{b})-I_{FIT}(\mathbf{b})}{I_{LSBX}}$$

for <0,0,…0> $\le$ **b** $\le$ <1,1,…1>
or equivalently for $0 \le k_b \le N\text{-}1$

$$INL_{\mathbf{k}}(\mathrm{b})=\frac{\sum_{i=1}^{n} b_i I_i-\frac{k_{\mathbf{b}}}{N-1}\sum_{i=1}^{n} I_i}{I_{LSBX}}$$

Current Steering DAC Statistical Characterization

Binary Weighted



Assume bundled current sources are comprised of unary current sources from same distribution

$$I_m = \sum_{k=2^{m-1}}^{2^m-1} I_{Gk} \qquad\qquad I_{Gk} = I_{NOM} + I_{RGk}$$

Thus

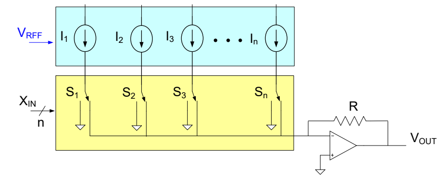
$$INL_{\mathbf{b}} = \frac{\displaystyle\sum_{i=1}^{n}\left(b_i\left(\sum_{k=2^{i-1}}^{2^i-1} I_{Gk}\right)\right) - \frac{k_{\mathbf{b}}}{N-1}\sum_{i=1}^{2^n-1} I_{Gi}}{I_{LSBX}}$$

Substituting the values for I_{Gk}, it can be shown that the nominal parts cancel thus

$$INL_{\mathbf{b}} = \frac{\displaystyle\sum_{i=1}^{n}\left(b_i\left(\sum_{k=2^{i-1}}^{2^i-1} I_{RGk}\right)\right) - \frac{k_{\mathbf{b}}}{N-1}\sum_{i=1}^{2^n-1} I_{RGi}}{I_{LSBX}}$$

Current Steering DAC Statistical Characterization

Binary Weighted



This can be expressed as

$$INL_{\mathbf{b}} = \sum_{i=1}^{n} \sum_{k=2^{i-1}}^{2^i-1} \left[b_i - \frac{k_{\mathbf{b}}}{N-1} \right] \frac{I_{RGk}}{I_{LSBX}}$$

This is now a sum of uncorrelated random variables, thus

$$\sigma_{INL_{\mathbf{b}}} = \sqrt{\sum_{i=1}^{n} \sum_{k=2^{i-1}}^{2^i-1} \left[b_i - \frac{k_{\mathbf{b}}}{N-1} \right]^2} \bullet \sigma_{\frac{I_{RGk}}{I_{LSBX}}}$$

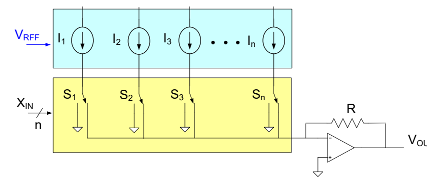
This reduces to

$$\sigma_{INL_{\mathbf{b}}} = \sqrt{\sum_{i=1}^{n} 2^{i-1} \left[b_i - \frac{k_{\mathbf{b}}}{N-1} \right]^2} \bullet \sigma_{\frac{I_{RGk}}{I_{LSBX}}}$$

Current Steering DAC Statistical Characterization

Binary Weighted

It can be shown that the maximum INL_b occurs at
b=<011…..11111> or b=<100….0000>



Substituting b=<1000….000>

$$\sigma_{INL_{\mathbf{b=<1000..0>}}} = \sqrt{2^{n-1}\left[1-\frac{N/2}{N-1}\right]^2 + \sum_{i=1}^{n-1} 2^{i-1}\left[\frac{N/2}{N-1}\right]^2} \bullet \sigma_{\frac{I_{RGk}}{I_{LSBX}}}$$

This simplifies to

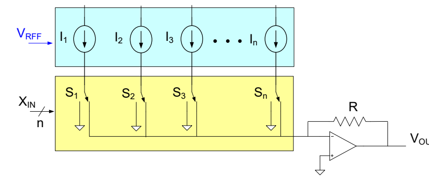
$$\sigma_{INL_{\mathbf{b=<1000..0>}}} = \sqrt{2^{n-1}\left[1-\frac{N/2}{N-1}\right]^2 + \sum_{i=1}^{n-1} 2^{i-1}\left[\frac{N/2}{N-1}\right]^2} \bullet \sigma_{\frac{I_{RGk}}{I_{LSBX}}}$$

This can be expressed as

$$\sigma_{INL_{\mathbf{b}}\mathbf{=<1000..0>}} = \sqrt{\frac{N}{2}\left[1-\frac{N/2}{N-1}\right]^2 + \left(\frac{N}{2}-1\right)\left[\frac{N/2}{N-1}\right]^2} \bullet \sigma_{\frac{I_{RGk}}{I_{LSBX}}}$$

Current Steering DAC Statistical Characterization

Binary Weighted



$$\sigma_{INL_{b=<1000..0>}} = \sqrt{\frac{N}{2}\left[1-\frac{N/2}{N-1}\right]^2 + \left(\frac{N}{2}-1\right)\left[\frac{N/2}{N-1}\right]^2} \bullet \sigma_{\frac{I_{RGk}}{I_{LSBX}}}$$

$$\sigma_{INL_{MAX}} \cong \sigma_{INL_{b=<1,0,...0>}} \cong \frac{\sqrt{N}}{2}\sigma_{\frac{I_{RG}}{I_{LSBX}}}$$

Note this is the same result as obtained for the unary DAC

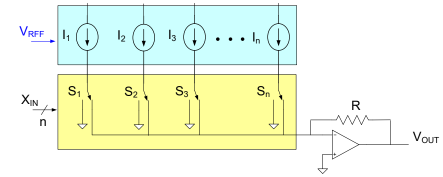
But closed form expressions do not exist for the INL of this DAC since the INL is an order statistic

Still need to obtain $\sigma_{\frac{I_{RG}}{I_{LSBX}}}$

Current Steering DAC Statistical Characterization

Unary Weighted and Binary Weighted

$$\sigma_{INL_{MAX}} \cong \sigma_{INL_{b=<1,0,...0>}} \cong \frac{\sqrt{N}}{2}\sigma_{\frac{I_{RG}}{I_{LSBX}}}$$



Note this is the same result as obtained for the unary DAC

Since INL is the max $|INK_k|$ is INL_{MAX} the same as INL? No!

Since $\sigma_{INL_{MAX}}$ is about the same for the Unary weighted structure and the Binary weighted structure, is the performance of both about the same?

No, DNL is much different !

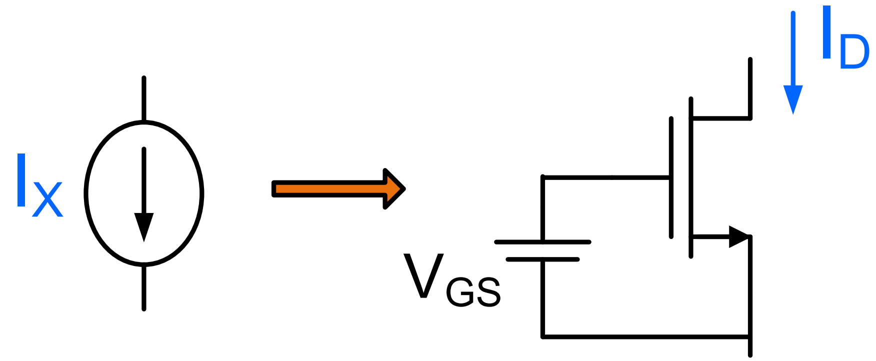
Instead of bundling unary current sources, could we simply take multiple outputs on a current mirror to generate the binary weighted currents ? This could be done in such a way that the area increases linearly rather than geometrically with the number of bits – much like with an R-2R DAC

Yes – but!

Will $\sigma_{INL_{MAX}} \cong \frac{\sqrt{N}}{2}\sigma_{\frac{I_{RG}}{I_{LSBX}}}$ also hold if we do not bundle unary current sources to obtain the binary current sources?

No, σ will be much different !

Statistical Modeling of Current Sources



Simple Square-Law MOSFET Model Usually Adequate for static Statistical Modeling

Assumption: Layout used to marginalize gradient effects, contact resistance and drain/source resistance neglected

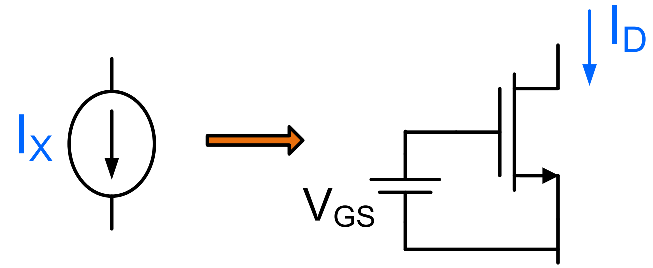
$$I_D = \frac{\mu C_{OX} W}{2L}\left(V_{GS} - V_{TH}\right)^2$$

Random Variables: $\{\mu, C_{OX}, V_{TH}, W, L\}$ Thus I_D is a random variable

From previous analysis, need: $\sigma_{\frac{I_D}{I_{DN}}}$

Statistical Modeling of Current Sources

$$I_D = \frac{\mu C_{OX} W}{2L}\left(V_{GS} - V_{TH}\right)^2$$



Random Variables: $\{\mu, C_{OX}, V_{TH}, W, L\}$ Thus I_D is a random variable

Will <u>assume</u> $\{\mu, C_{OX}, V_{TH}, W, L\}$ are uncorrelated

This is not true : T_{OX} is a random variable that affects both V_{TH} and C_{OX}

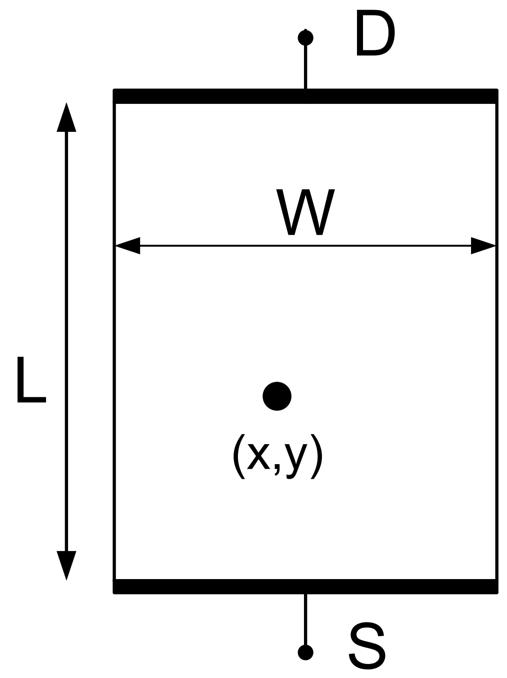
- This assumption is widely used and popularized by Pelgrom
- It is also implicit in the statistical model available in simulators such as SPECTRE
- Statistical information about T_{OX} often not available
- Drenen and McAndrew (NXP) published several papers that point out limitations
- Would be better to model physical parameters rather than model parameters but more complicated
- Statistical analysis tools at NXP probably have this right but not widely available
- Assumption simplifies analysis considerably
- Error from neglecting correlation is usually quite small but don't know how small

Statistical Modeling of Current Sources

Model parameters are position dependent

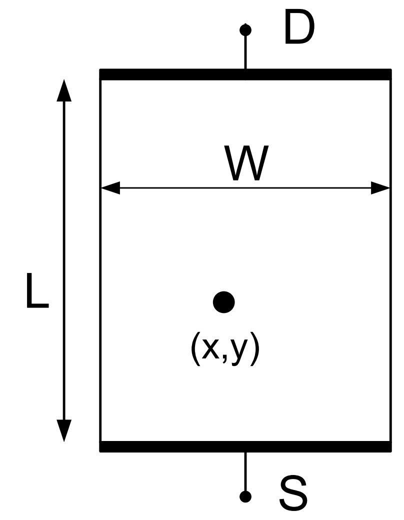
$$I_D = \frac{\mu C_{OX} W}{2L}\left(V_{GS} - V_{TH}\right)^2$$

$\mu(x,y)$, $C_{OX}(x,y)$, $V_{TH}(x,y)$



Statistical Modeling of Current Sources

Model parameters are position dependent



$$I_D = \frac{\mu C_{OX} W}{2L}\left(V_{GS} - V_{TH}\right)^2$$

Assume that model parameters can be modeled as a position-weighted integral

$$\mu = \frac{\int_A \mu(x,y)\,dxdy}{A}$$

$$C_{OX} = \frac{\int_A C_{OX}(x,y)\,dxdy}{A}$$

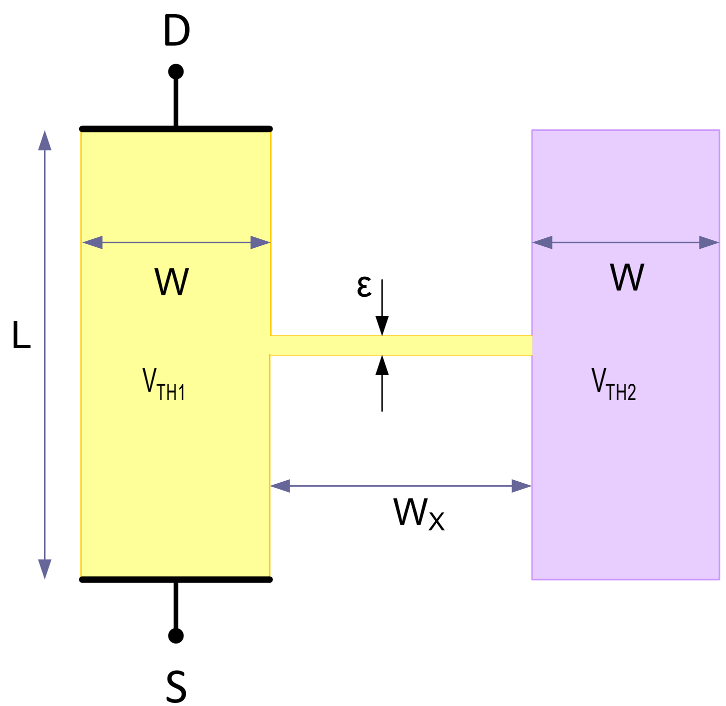
$$V_{TH} = \frac{\int_A V_{TH}(x,y)\,dxdy}{A}$$

Reasonably good assumption if current density is constant

Statistical Modeling of Current Sources

Assume that model parameters can be modeled as a position-weighted integral

As seen for resistors, this model is not good if current density is not constant



$$I_D \simeq \frac{\mu C_{OX} W}{2L}\left(V_{GS} - V_{TH1}\right)^2$$

$$V_{THEQ} = \frac{\int_A V_{TH}(x,y)\,dxdy}{A} \simeq \frac{V_{TH1}+V_{TH2}}{2}$$

If V_{TH1}=1V, V_{TH2}=2V

V_{THEQ}=1.5V

Note dramatically different current densities

But reasonably good assumption if current density is constant

Statistical Modeling of Current Sources

$$I_D = \frac{\mu C_{OX} W}{2L}\left(V_{GS} - V_{TH}\right)^2$$

Model parameters characterized by following equations

$$\mu = \mu_N + \mu_R$$

$$V_{TH} = V_{THN} + V_{THR}$$

$$C_{OX} = C_{OXN} + C_{OXR}$$

$$L = L_N + L_R$$

$$W = W_N + W_R$$

Neglecting random part of W and L which are usually less important

$$I_D = \frac{\left(\mu_N + \mu_R\right)\left(C_{OXN} + C_{OXR}\right) W}{2L}\left(V_{GS} - V_{THN} - V_{THR}\right)^2$$

Statistical Modeling of Current Sources

$$I_D = \frac{(\mu_N + \mu_R)(C_{OXN} + C_{OXR})W}{2L}(V_{GS} - V_{THN} - V_{THR})^2$$

This appears to be a highly nonlinear function of random variables !!

Will now linearize the relationship between I_D and the random variables

Since the random variables are small, we can do a Taylor's series expansion and truncate after first-order terms to obtain

$$I_D \cong \frac{\mu_N C_{OXN} W}{2L}(V_{GS} - V_{THN})^2 + \mu_R \frac{C_{OXN} W}{2L}(V_{GS} - V_{THN})^2 + C_{OXR}\frac{\mu_N W}{2L}(V_{GS} - V_{THN})^2 - V_{THR}\frac{\mu_N C_{OXN} W}{L}(V_{GS} - V_{THN})$$

This is a linearization of I_D in the random variables μ_R, C_{OXR}, and V_{THR}

$$I_{DR} \cong \mu_R \frac{C_{OXN} W}{2L}(V_{GS} - V_{THN})^2 + C_{OXR}\frac{\mu_N W}{2L}(V_{GS} - V_{THN})^2 - V_{THR}\frac{\mu_N C_{OXN} W}{L}(V_{GS} - V_{THN})$$

$$\frac{I_{DR}}{I_{DN}} \cong \mu_R \frac{\frac{C_{OXN} W}{2L}(V_{GS} - V_{THN})^2}{I_{DN}} + C_{OXR}\frac{\frac{\mu_N W}{2L}(V_{GS} - V_{THN})^2}{I_{DN}} - V_{THR}\frac{\frac{\mu_N C_{OXN} W}{L}(V_{GS} - V_{THN})}{I_{DN}}$$

Could easily include L_R and W_R but usually not important unless lots of perimeter

Statistical Modeling of Current Sources

$$\frac{I_{DR}}{I_{DN}} \cong \mu_R \frac{\frac{C_{OXN}W}{2L}\left(V_{GS}\text{-}V_{THN}\right)^2}{I_{DN}} + C_{OXR}\frac{\frac{\mu_N W}{2L}\left(V_{GS}\text{-}V_{THN}\right)^2}{I_{DN}} - V_{THR}\frac{\frac{\mu_N C_{OXN} W}{L}\left(V_{GS}\text{-}V_{THN}\right)}{I_{DN}}$$

$$I_{DN} = \frac{\mu_N C_{OXN} W}{2L}\left(V_{GS}\text{-}V_{THN}\right)^2$$

$$\frac{I_{DR}}{I_{DN}} \cong \mu_R \frac{\frac{C_{OXN}W}{2L}\left(V_{GS}\text{-}V_{THN}\right)^2}{\frac{\mu_N C_{OXN} W}{2L}\left(V_{GS}\text{-}V_{THN}\right)^2} + C_{OXR}\frac{\frac{\mu_N W}{2L}\left(V_{GS}\text{-}V_{THN}\right)^2}{I\,\frac{\mu_N C_{OXN} W}{2L}\left(V_{GS}\text{-}V_{THN}\right)^2} - V_{THR}\frac{\frac{\mu_N C_{OXN} W}{L}\left(V_{GS}\text{-}V_{THN}\right)}{\frac{\mu_N C_{OXN} W}{2L}\left(V_{GS}\text{-}V_{THN}\right)^2}$$

$$\frac{I_{DR}}{I_{DN}} \cong \frac{\mu_R}{\mu_N} + \frac{C_{OXR}}{C_{OXN}} - \frac{2V_{THR}}{\left(V_{GS}\text{-}V_{THN}\right)}$$

Thus

$$\sigma_{\frac{I_{DR}}{I_{DN}}} = \sqrt{\sigma^2_{\frac{\mu_R}{\mu_N}} + \sigma^2_{\frac{C_{OXR}}{C_{OXN}}} + 4\left(\frac{V_{THN}}{V_{GS}-V_{THN}}\right)^2 \sigma^2_{\frac{V_{THR}}{V_{THN}}}} \quad \text{or} \quad \sigma_{\frac{I_{DR}}{I_{DN}}} = \sqrt{\sigma^2_{\frac{\mu_R}{\mu_N}} + \sigma^2_{\frac{C_{OXR}}{C_{OXN}}} + \left(\frac{2}{V_{GS}-V_{THN}}\right)^2 \sigma^2_{V_{THR}}}$$

Statistical Modeling of Current Sources

$$\sigma_{\frac{I_{DR}}{I_{DN}}} = \sqrt{\sigma^2_{\frac{\mu_R}{\mu_N}} + \sigma^2_{\frac{C_{OXR}}{C_{OXN}}} + 4\left(\frac{V_{THN}}{V_{GS}-V_{THN}}\right)^2 \sigma^2_{\frac{V_{THR}}{V_{THN}}}} \quad \text{or} \quad \sigma_{\frac{I_{DR}}{I_{DN}}} = \sqrt{\sigma^2_{\frac{\mu_R}{\mu_N}} + \sigma^2_{\frac{C_{OXR}}{C_{OXN}}} + \left(\frac{2}{V_{GS}-V_{THN}}\right)^2 \sigma^2_{V_{THR}}}$$

It will be assumed that
(will discuss assumption later)

$$\sigma^2_{\frac{\mu_R}{\mu_N}} = \frac{A_\mu^2}{WL}$$

$$\sigma^2_{\frac{C_{OXR}}{C_{OXN}}} = \frac{A_{Cox}^2}{WL}$$

$$\sigma^2_{V_{THR}} = \frac{A_{VT0}^2}{WL}$$

where A_μ,A_{Cox},A_{VT0} are Pelgrom process parameters

$$\sigma_{\frac{I_{DR}}{I_{DN}}} = \frac{1}{\sqrt{WL}}\sqrt{A_\mu^2 + A_{Cox}^2 + \frac{4}{V_{EB}^2}A_{VT0}^2}$$

Define

$$A_\beta = \sqrt{A_\mu^2 + A_{Cox}^2}$$

Thus

$$\sigma_{\frac{I_{DR}}{I_{DN}}} = \frac{1}{\sqrt{WL}}\sqrt{A_\beta^2 + \frac{4}{V_{EB}^2}A_{VT0}^2}$$

Often only A_β is available

Statistical Modeling of Current Sources

$$\sigma_{\frac{I_{DR}}{I_{DN}}} = \frac{1}{\sqrt{WL}}\sqrt{A_{\beta}^2 + \frac{4}{V_{EB}^2}A_{VT0}^2}$$

Gate area: A=WL

- Standard deviation decreases with $\sqrt{A}$
- Large V_{EB} reduces standard deviation
- Operating near cutoff results in large mismatch
- Often threshold voltage variations dominate mismatch

$$\sigma_{\frac{I_{DR}}{I_{DN}}} \cong \frac{2}{V_{EB}\sqrt{WL}}A_{VT0}$$

Stay Safe and Stay Healthy !

End of Lecture 10